# Rational homotopy theory and algebraic models

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# Outline

1. Introduction of rational homotopy theory and algebraic models
2. Sullivan models
3. Realization of minimal Sullivan algebras
4. Formality of sphere bundles

# Equivalence relations between spaces

Homotopy equivalent

$\Downarrow$

Weak homotopy equivalent — **Model:** CW complex

$\Downarrow$

Rational homotopy equivalent — Connected by maps preserving $H^*(-;\mathbb{Q})$ (also preserving $\pi_* \otimes \mathbb{Q}$ if 1-connected). **Model:** CDGA

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Same Euler number — **Model:** integer

**CDGA:** commutative differential graded algebra
Commutative means **graded commutative**, i.e.

$$x \cdot y = (-1)^{\deg x \deg y} y \cdot x.$$

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# Model of spaces

| Space | Model |
|---|---|
| Smooth manifold $M$ (ground field $\mathbb{R}$) | $\Omega^*(M)$ |
| General topological space $X$ | $A_{PL}(X)$ |

$\Omega^*(M)$ : The CDGA of differential forms on $M$.
$A_{PL}(X)$ : The CDGA of **polynomial differential forms**, which are simplicial maps $S_*(X) \to A_{PL}$.

$S_*(X)$ : Simplicial set of singular simplicies.
$(A_{PL})_n = \Lambda(t_0, \ldots, t_n, dt_0, \ldots, dt_n)/\sim$,
where $\deg t_i = 0$, $\deg dt_i = 1$,
$t_0 + \ldots + t_n \sim 1, \quad dt_0 + \ldots + dt_n \sim 0.$
$\Lambda$ : free (graded) commutative algebra (usually with ground field $\mathbb{Q}$).
e.g. $\Lambda(t_0, dt_0) = \langle 1, t_0, t_0^2, \ldots, dt_0, t_0 dt_0, t_0^2 dt_0, \ldots \rangle$

The simplicial set structure on $A_{PL}$ is given by

$$\partial_i : (A_{PL})_n \to (A_{PL})_{n-1}, \quad t_k \mapsto \begin{cases} t_k, & k < i \\ 0, & k = i \\ t_{k-1}, & k > i \end{cases}$$

$$s_j : (A_{PL})_n \to (A_{PL})_{n+1}, \quad t_k \mapsto \begin{cases} t_k, & k < j \\ t_k + t_{k+1}, & k = j \\ t_{k+1}, & k > j \end{cases}$$

The CDGA structure on $A_{PL}(X)$ is induced by the CDGA structure on each $(A_{PL})_n$.

# Definition of model

For a smooth manifold $M$, $\Omega^*(M)$ and $A_{PL}(M) \otimes \mathbb{R}$ are **equivalent**, in the sense that they can be connected by **quasi-isomorphisms** (CDGA morphisms inducing isomorphisms on cohomology).

**Theorem**

*There exists a quasi-isomorphism of cochain complexes* $\oint : A_{PL}(X) \to C^*(X; \mathbb{Q})$.

**Idea of proof.**

Taking integrals of polynomial differential forms on suitable simplexes. Moreover, $A_{PL}(X)$ and $C^*(X; \mathbb{Q})$ can be connected by quasi-isomorphisms (non-commutative) DGAs.

(See Section 10(d), (e) of *Rational Homotopy Theory* by Félix, Halperin and Thomas.)

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Any CDGA equivalent to $A_{PL}(X)$ is called a **model** of $X$.

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# Minimal Sullivan algebra

## Definition

A **Sullivan algebra** $(\Lambda V, d)$ is a free commutative graded algebra $\Lambda V$, generated by a graded vector space $V$ of positive degree with a filtration

$$0 = V(-1) \subset V(0) \subset V(1) \subset \ldots \subset V(n) \subset \ldots \subset V = \bigcup_{n=0}^{\infty} V(n)$$

together with a differential $d$ satisfying

$$dV(n) \subset \Lambda V(n-1).$$

If in addition that $dV(n) \subset \Lambda^{\geq 2} V(n-1)$, where RHS is spanned by elements in $\Lambda V(n-1)$ of wordlength at least 2, we say $\Lambda V$ is **minimal**.

We will write $\Lambda V$ short for $(\Lambda V, d)$.

# Minimal Sullivan model

**Definition**

For every *connected* ($H^0$ is ground field) CDGA $A$, there exists a quasi-isomorphism $\Lambda V \to A$ from a minimal Sullivan algebra $\Lambda V$, unique up to isomorphism. $\Lambda V$ is called the **minimal Sullivan model** of $A$, and, if $A = A_{PL}(X)$, is also called the **minimal Sullivan model** of the space $X$.

**Idea of proving existence.**
Construct $V^n$ inductively, such that $\Lambda V^{\leq n} \to A_{PL}(X)$ induces isomorphisms on $H^{\leq n}$ and is injective on $H^{n+1}$. Here $\Lambda V^{\leq n}$ is the Sullivan algebra generated by $V^{\leq n}$.

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# Constructing minimal Sullivan model

For simplicity, we assume that $X$ is 1-connected and $H^*(X)$ is of finite type.

1. Set $V^2 \cong H^2(X)$, and construct a linear map sending $V^2$ to the representatives in $A_{PL}(X)$.
   This induces a CDGA morphism $\Lambda V^2 \to A_{PL}(X)$, which is isomorphic on $H^2$. As $(\Lambda V^2)^3 = 0$, it is injective on $H^3$.
2. Set $C^3 \cong H^3(X)$, $N^3 \cong \ker(H^4(\Lambda V^2) \to H^4(X))$ under $d$, and $V^3 = C^3 \oplus N^3$.
   Extend $\Lambda V^2 \to A_{PL}(X)$ to $\Lambda V^{\leq 3}$ by sending $C^3$ to the representatives and making the image of $N^3$ compatible with $d$.
   This makes that $\Lambda V^{\leq 3} \to A_{PL}(X)$ is isomorphic on $H^{\leq 3}$ and injective on $H^4$.
3. Set $C^4$ satisfying $H^4(\Lambda V^{\leq 3}) \oplus C^4 \cong H^4(X)$, $N^4 \cong \ker(H^5(\Lambda V^{\leq 3}) \to H^5(X))$ under $d$, and $V^4 = C^4 \oplus N^4$.

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# Examples

$S^{2n+1}$: $\Lambda(x),\ \deg x = 2n+1,\ dx = 0.$

$S^{2n}$: $\Lambda(x, y),\ \deg x = 2n,\ \deg y = 4n-1,\ dx = 0,\ dy = x^2.$

$S^3 \vee S^3$:

First need $x, y$ of degree 3, such that $dx = dy = 0$.
Next need $z$ of degree 5 such that $dz = xy$.
Then need $u, v$ of degree 7 such that $du = xz, dv = yz$...... The vector space $V$ generating the minimal Sullivan model is infinite dimensional, but is of finite type.

$S^1 \vee S^1$:

The model is same as the model for $S^3 \vee S^3$ except the degree. Here $\deg x = \deg y = \deg z = \deg u = \deg v = \ldots = 1$. In particular, $\dim V^1 = \infty$.

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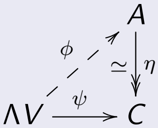
# Lifting lemma for Sullivan algebras

**Lemma (Lifting Lemma)**

*Let $\eta : A \to C$ be a surjective quasi-isomorphism of CDGA, and $\psi : \Lambda V \to C$ be a CDGA morphism from a Sullivan algebra. Then there exists a CDGA morphism $\phi : \Lambda V \to A$ such that $\eta \circ \phi = \psi$.*




**Idea of proof.** Construct $\phi$ on V(k) inductively.

# Homotopy of CDGA morphisms from Sullivan algebras

**Definition**

Let $\phi_0, \phi_1 : \Lambda V \to A$ be CDGA morphisms from some Sullivan algebra $\Lambda V$. We say $\phi_0$ and $\phi_1$ are **homotopic** if there exists a morphism $\Phi : \Lambda V \to A \otimes \Lambda(t, dt)$ with $\deg t = 0$ such that $(id_A \otimes \epsilon_i) \circ \Phi = \phi_i$ for $i = 0, 1$. Here $\epsilon_i : \Lambda(t, dt) \to \mathbb{Q}$ sends $t$ to $i$. When $\phi_0$ and $\phi_1$ are homotopic, we denote it as $\phi_0 \sim \phi_1$.

**Proposition**

*Suppose that $f_0, f_1 : X \to Y$ are homotopic maps on topological spaces. Let $\psi : \Lambda V \to A_{PL}(Y)$ be a CDGA morphism from a Sullivan algebra. Then $A_{PL}(f_0) \circ \psi \sim A_{PL}(f_1) \circ \psi : \Lambda V \to A_{PL}(X)$.*

**Idea of proof.** $\Lambda(t, dt) \to A_{PL}(I)$ is an injective quasi-isomorphism.

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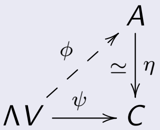
# Lifting lemma up to homotopy

## Lemma

*Let $\eta : A \to C$ be a quasi-isomorphism of CDGA, and $\psi : \Lambda V \to C$ be a CDGA morphism from a Sullivan algebra. Then there exists a CDGA morphism $\phi : \Lambda V \to A$ such that $\eta \circ \phi \sim \psi$, i.e. the diagram below is commutative up to homotopy.*




*Actually, $\eta$ induces a bijection between the homotopy classes $[\Lambda V, A]$ and $[\Lambda V, C]$.*

**Idea of proof.**

1. First consider the case that $\eta$ is surjective. Then the surjectivity of $[\Lambda V, A] \to [\Lambda V, C]$ follows from the lifting lemma.
   For the injectivity, we can lift a homotopy $\Lambda V \to C \otimes \Lambda(t, dt)$ to a homotopy $\Lambda V \to A \otimes \Lambda(t, dt)$ by a suitable construction.
2. For general $\eta$, set $E(C) = \Lambda(C' \oplus dC')$, where $C'$ is isomorphic to $C$ as a graded vector space, and $d : C' \xrightarrow{\cong} dC'$.
   Let $\rho : E(C) \to C$ be the natural morphism induced by the isomorphism $C' \to C$. Then $\eta$ can be factored as

$$A \overset{\simeq}{\hookrightarrow} A \otimes E(C) \underset{\simeq}{\overset{\eta \cdot \rho}{\twoheadrightarrow}} C.$$

   The inclusion $A \xrightarrow{\sim} A \otimes E(C)$ has a left inverse, which is a surjective quasi-isomorphism. Applying the previous case on it and $\eta \cdot \rho$, we have obtained

$$[\Lambda V, A] \cong [\Lambda V, A \otimes E(C)] \cong [\Lambda V, C].$$

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# Homotopic morphisms are same on homology

## Proposition

1. *If* $\phi_0 \sim \phi_1 : \Lambda V \to A$*, then* $H(\phi_0) = H(\phi_1)$.
2. *If* $\phi_0 \sim \phi_1 : \Lambda V \to \Lambda W$*, then* $H(Q(\phi_0)) = H(Q(\phi_1))$*. Here* $Q(\Lambda V) = \frac{(\Lambda V)^+}{(\Lambda V)^+ \cdot (\Lambda V)^+}$ *and it is isomorphic to* $V$ *as a graded vector space.* $Q(\phi_0), Q(\phi_1) : Q(\Lambda V) \to Q(\Lambda W)$ *are induced by* $\phi_0$ *and* $\phi_1$.

**Idea of proof.**

1. Construct a chain homotopy map $h : \Lambda V \to A$ such that $\phi_1 - \phi_0 = dh + hd$.
2. A homotopy $\Phi$ for $\phi_0 \sim \phi_1$ induces a morphism $\overline{\Phi} : Q(\Lambda V) \to Q(\Lambda W) \otimes \Lambda(t, dt)$.
   If in addition that $\Lambda W$ is minimal, the cocycles in the codomain are in $Q(\Lambda W) \otimes (\mathbb{Q} \oplus \Lambda(t)dt)$. Then

$$(id_{Q(\Lambda W)} \otimes \epsilon_0) \circ \overline{\Phi} = (id_{Q(\Lambda W)} \otimes \epsilon_1) \circ \overline{\Phi}.$$

   Finally discuss the general case.

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# Uniqueness of minimal Sullivan model

## Proposition

*If $\phi : \Lambda V \to \Lambda W$ is a quasi-isomorphism of minimal Sullivan algebras, then it is an isomorphism.*

**Idea of proof.**

Lift $id : \Lambda W \to \Lambda W$ through $\phi$, we have obtained a morphism $\psi : \Lambda W \to \Lambda V$ such that $\psi\phi \sim id$.




Then $H(Q(\phi) \circ Q(\psi)) = id$. As $\Lambda V$ and $\Lambda W$ are minimal, $H(Q(\Lambda V)) \cong V$ and $H(Q(\Lambda W)) \cong W \implies Q(\phi) \circ Q(\psi) = id$ on $W \implies \phi\psi = id$ on $\Lambda W$. Finally, a same discussion on $\psi$ shows that it has a right inverse.

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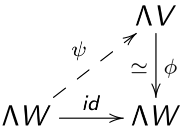


Then $H(Q(\phi) \circ Q(\psi)) = id$. As $\Lambda V$ and $\Lambda W$ are minimal, $H(Q(\Lambda V)) \cong V$ and $H(Q(\Lambda W)) \cong W \implies Q(\phi) \circ Q(\psi) = id$ on $W \implies \phi\psi = id$ on $\Lambda W$. Finally, a same discussion on $\psi$ shows that it has a right inverse.

# Sullivan representatives

## Proposition

*Let $f : X \to Y$ be a continuous map of path-connected spaces, and $\Lambda V, \Lambda W$ be the minimal Sullivan models of $X$ and $Y$ respectively. Then there exists a unique morphism $\phi : \Lambda W \to \Lambda V$ making the diagram below commutative up to homotopy.*

$$\begin{array}{ccc}
\Lambda W & \overset{\phi}{\dashrightarrow} & \Lambda V \\
\downarrow{\scriptstyle \simeq} & & \downarrow{\scriptstyle \simeq} \\
A_{PL}(Y) & \xrightarrow{A_{PL}(f)} & A_{PL}(X)
\end{array}$$

# Λ-extension and Relative Sullivan algebra

## Definition

Let $B$ be a connected CDGA. A **Λ-extension** of $B$ is an inclusion $B \to B \otimes \Lambda Z$ such that $\deg Z \geq 0$ and $Z$ is the union of an increasing sequence of subspaces

$$Z(0) \subset Z(1) \subset \ldots \subset Z(k) \subset \ldots$$

satisfying

$$dZ(k) \subset (\mathbb{Q} \oplus B^{\geq 1}) \otimes \Lambda Z(k-1).$$

If in addition $\deg Z \geq 1$, $B \to B \otimes \Lambda Z$ is called a **Sullivan extension**, and $B \otimes \Lambda Z$ is called a **relative Sullivan algebra**.

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# Factorization theorem

Let $B \otimes \Lambda Z$ be a relative Sullivan algebra. $B$ is called the **base algebra** of $B \otimes \Lambda Z$.
On the other hand, an augmentation $B \to \mathbb{Q}$ induces a morphism $B \otimes \Lambda Z \to \Lambda Z$, which gives $\Lambda Z$ a Sullivan algebra structure. This $\Lambda Z$ is called the **fiber algebra** of $B \otimes \Lambda Z$.
If $\Lambda Z$ is minimal, we call $B \otimes \Lambda Z$ a **minimal relative algebra**.

**Remark.**
We can also define the minimality for $\Lambda$-extensions similarly.

**Theorem**

*Let $f : B \to C$ be a morphism of connected CDGAs. Then it can be factored as a minimal $\Lambda$-extension $B \to B \otimes \Lambda Z$ and a quasi-isomorphism $B \otimes \Lambda Z \to C$ uniquely up to isomorphism.*
*Moreover, if $f^* : H^1(B) \to H^1(C)$ is injective, then $B \otimes \Lambda W$ is a relative Sullivan algebra, which is called the Sullivan model of $f$.*

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# Fibration and relative Sullivan algebra

## Theorem

*Suppose that $F \to E \to B$ is a fibration, or a Serre fibration with $F$ being a CW complex, satisfying the following conditions.*
*(1) $F$, $E$ and $B$ are path-connected.*
*(2) $\pi_1(B)$ acts on $H^*(F;\mathbb{Q})$ locally nilpotently.*
*(3) One of $H^*(F)$ is of $H^*(B)$ has finite type.*
*(4) There is a relative Sullivan algebra $\Lambda V \otimes \Lambda Z$ and morphisms making the following diagram commutative.*

$$\begin{array}{ccccc}
(\Lambda V, d) & \hookrightarrow & (\Lambda V \otimes \Lambda Z, d) & \xrightarrow{pr} & (\Lambda Z, \bar{d}) \\
\downarrow f_B & & \downarrow f_E & & \downarrow f_F \\
A_{PL}(B) & \longrightarrow & A_{PL}(E) & \longrightarrow & A_{PL}(F)
\end{array}$$

*(i) If $f_B$ are $f_E$ are both quasi-isomorphisms, then so is $f_F$.*
*(ii) If $f_B$ are $f_F$ are both quasi-isomorphisms, then so is $f_E$.*

# A counterexample

The requirement that $\pi_1(Y)$ acts on $H^*(F;\mathbb{Q})$ locally nilpotently is necessary.

## Example

Let $F$ be the homotopy fiber of $S^1 \vee S^2 \to S^1$ contracting $S^2$ to a point. Then we have a fibration $F \to E \to B$, with $F \simeq \widetilde{S^1 \vee S^2}$, $E \simeq S^1 \vee S^2$ and $B = S^1$. The $\pi_1(B)$-action is not locally nilpotent. Also $\dim H_2(F) = \infty$, then $\dim H^2(F)$ is uncountable.

On the other hand, $\Lambda v$ is a minimal Sullivan model of $B$ with $\deg v = 1$. It can be extended to a minimal Sullivan model $\Lambda v \otimes \Lambda Z$ with $\deg Z \geq 2$ and $\dim Z^2$ countable. But the induced $\Lambda Z \to A_{PL}(F)$ is not a quasi-isomorphism, because the dimension of $H^2$ are different.

# Minimal Sullivan model and rational homotopy group

**Theorem**

*Suppose that $X$ is simply-connected and $H^*(X;\mathbb{Q})$ has finite type. $\Lambda V$ is the minimal Sullivan model of $X$. Then*

$$V^n \cong Hom(\pi_n(X), \mathbb{Q}).$$

**Idea of proof.**

1. Start from $X = K(\pi, 1)$ with $\pi$ abelian and of finite rank $r$.
Let $a_1, \ldots, a_r \in \pi$ be a basis of $\pi \otimes \mathbb{Q}$. They give a morphism $\mathbb{Z}^r \to \pi$, then a map $K(\mathbb{Z}^r, 2) \to K(\pi, 2)$ which is isomorphism on $\pi_* \otimes \mathbb{Q}$.

### Theorem (Whitehead-Serre)

*Suppose that $f : Y \to Z$ is a continuous map between 1-connected spaces. Then the following statements are equivalent.*
*(i) $\pi_*(f) \otimes \mathbb{Q} : \pi_*(Y) \otimes \mathbb{Q} \to \pi_*(Z) \otimes \mathbb{Q}$ is an isomorphism.*
*(ii) $H_*(f; \mathbb{Q}) : H_*(Y; \mathbb{Q}) \to H_*(Z; \mathbb{Q})$ is an isomorphism.*
*(iii) $H_*(\Omega f; \mathbb{Q}) : H_*(\Omega Y; \mathbb{Q}) \to H_*(\Omega Z; \mathbb{Q})$ is an isomorphism.*

Thus,

$$\begin{aligned} H^*(T^r; \mathbb{Q}) = H^*(K(\mathbb{Z}^r, 1); \mathbb{Q}) &= H^*(\Omega K(\mathbb{Z}^r, 2); \mathbb{Q}) \\ &\cong H^*(\Omega K(\pi, 2); \mathbb{Q}) = H^*(X; \mathbb{Q}). \end{aligned}$$

The minimal Sullivan model of $T^r$ can be taken as $\Lambda(x_1, \ldots, x_r)$ with all $\deg x_i = 1$ and $dx_i = 0$.

**Remark.** The Whitehead-Serre theorem may not hold if the spaces are not 1-connected. That is why we prove the statement in this way instead of constructing $K(\mathbb{Z}^r, 1) \to K(\pi, 1)$ directly.

2. Use induction. Suppose the statement holds for $K(\pi, k)$ for $k < n$. Let $X = K(\pi, n)$.
Suppose that $\Lambda V$ is a minimal model of $X$. Then $V$ is concentrated in degree $\geq n$. Set $U$ such that $d : U^{k-1} \xrightarrow{\cong} V^k$ for all $k$. There exists the following commutative diagram.

$$
\begin{array}{ccccc}
(\Lambda V, d) & \hookrightarrow & (\Lambda V \otimes \Lambda U, d) & \xrightarrow{pr} & (\Lambda U, \bar{d}) \\
\simeq \downarrow & & \simeq \downarrow & & \downarrow \\
A_{PL}(X) & \longrightarrow & A_{PL}(PX) & \longrightarrow & A_{PL}(\Omega X)
\end{array}
$$

Then $\Lambda U$ is a minimal Sullivan model of $\Omega X = K(\pi, n-1)$.
By inductive hypothesis $U = U^{n-1}$ and $\dim U = \dim(\pi \otimes \mathbb{Q})$. So $V = V^n$ and $\dim V = \dim(\pi \otimes \mathbb{Q})$.

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3. Consider general $X$. Prove the statement holds for all $X_n$ on the Postnikov tower inductively.

$$
\begin{array}{ccc}
 & & \cdots \\
 & & \downarrow \\
K(\pi_4(X), 4) & \longrightarrow & X_4 \\
 & & \downarrow \\
K(\pi_3(X), 3) & \longrightarrow & X_3 \\
 & & \downarrow \\
 & & X_2 = K(\pi_3(X), 2)
\end{array}
$$

4. For each $n$, turn $X \to X_n$ into a fibration $F_n \to X \to X_n$. Take a minimal Sullivan model $\Lambda W$ of $X_n$, and extend it to a minimal relative Sullivan algebra $\Lambda W \otimes \Lambda U$ which is a model of $X$.

$$\begin{array}{ccccc} (\Lambda W, d) & \hookrightarrow & (\Lambda W \otimes \Lambda U, d) & \xrightarrow{pr} & (\Lambda U, \bar{d}) \\ \simeq\downarrow & & \simeq\downarrow & & \downarrow \\ A_{PL}(X_n) & \longrightarrow & A_{PL}(X) & \longrightarrow & A_{PL}(F_n) \end{array}$$

Then $\Lambda U$ is a minimal Sullivan model of the $n$-connected space $F_n$. So $\deg U \geq n+1$ and for degree reason $\Lambda W \otimes \Lambda U$ is a minimal Sullivan algebra (i.e. the minimal Sullivan model of $X$). Therefore,

$$(W \oplus U)^n = W^n \cong \mathrm{Hom}(\pi_n(X_n), \mathbb{Q}) = \mathrm{Hom}(\pi_n(X), \mathbb{Q}).$$

# A counterexample

For non-simply-connected spaces, its rational homotopy group may not be represented by the minimal Sullivan model.

## Example

Let $X = \mathbb{R}P^2$. Then $H^n(X;\mathbb{Z}) = \begin{cases} \mathbb{Z}, & n = 0 \\ \mathbb{Z}/2\mathbb{Z}, & n = 2 \\ 0, & \text{otherwise} \end{cases}$

So $H^*(X;\mathbb{Q}) = \mathbb{Q}$ and its minimal Sullivan model $\Lambda V$ is $\mathbb{Q}$ trivially ($V = 0$). But $\pi_2(X) \otimes \mathbb{Q} = \mathbb{Q}$ is non-trivial.

# Homotopy group of minimal Sullivan algebra

**Definition**

Let $\Lambda V$ be a minimal Sullivan algebra. $\mathrm{Hom}(V^n, \mathbb{Q})$ is called the $n$-th **homotopy group** ($n \geq 2$) of $\Lambda V$, and written as $\pi_n(\Lambda V)$.

**Remark.** $\pi_1(\Lambda V)$ is defined differently, and is non-abelian in general. It is only well-defined when $\dim H^1(\Lambda V) < \infty$.

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# Homotopy Lie algebra of $\Lambda V$

Actually, $\mathrm{Hom}(V, \mathbb{Q})$ has a Lie algebra structure, and is called the **homotopy Lie algebra** of $\Lambda V$. Set $L_{n-1} = \mathrm{Hom}(V^n, \mathbb{Q})$ for $n \geq 1$. This gives a pairing $V \times L \to \mathbb{Q}$ by

$$\langle v, x \rangle = (-1)^{degv} x(v).$$

(More precisely, this pairing is $\langle v, sx \rangle$, where $sx$ is the suspension of $x$.)

This paring can be extended to $\Lambda^p V \times L^p \to \mathbb{Q}$ (Here $\Lambda^p V$ is the subspace of $V$ spanned by elements of wordlength $p$ and $L^p = L \times \ldots \times L$) as

$$\langle v_1 \wedge \ldots \wedge v_p, x_p, \ldots, x_1 \rangle = \sum_{\sigma \in S_p} \epsilon_\sigma \langle v_{\sigma(1)}, x_1 \rangle \ldots \langle v_{\sigma(p)}, x_p \rangle.$$

Let $d_1 v$ denote the component of $dv$ in $\Lambda^2 V$. The Lie bracket of $L$ is given by

$$\langle v, [x, y] \rangle = (-1)^{degy+1} \langle d_1 v, x, y \rangle.$$

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# The definition of $\pi_1(\Lambda V)$

$\pi_1(\Lambda V)$ is defined as $\exp L_0$. The precise definition is given as follows.

$UL_0$: The universal enveloping algebra of $L_0$, i.e. $TL_0/\sim$ with $x \otimes y - y \otimes x \sim [x, y]$.

$I_{L_0}$: The ideal in $UL_0$ generated by $L_0$.

$\widehat{UL_0} := \varprojlim_n UL_0/I_{L_0}^n$, the completion of $UL_0$.

$\widehat{I_{L_0}} := \varprojlim_n I_{L_0}/I_{L_0}^n$.

**Lemma.** There exist inverse bijections

$$\widehat{I_{L_0}} \underset{\log}{\overset{\exp}{\rightleftarrows}} 1 + \widehat{I_{L_0}},$$

where

$$\exp x = \sum_{n=0}^{\infty} \frac{1}{n!} x^n, \text{ and } \log(1+x) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} x^n.$$

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$\Delta$: $L_0 \to L_0 \oplus L_0$, $x \mapsto (x,0)+(0,x)$, the diagonal map.
$U\Delta$: $UL_0 \to U(L_0 \oplus L_0) = UL_0 \otimes UL_0$, induced by $\Delta$.
$\widehat{U\Delta}$: $\widehat{UL_0} \to \widehat{UL_0}\widehat{\otimes}\widehat{UL_0}$, completion of $U\Delta$.
$P_{L_0} := \{x \in \widehat{I}_{L_0} | \widehat{U\Delta}(x) = x\widehat{\otimes}1 + 1\widehat{\otimes}x\}$.
$G_{L_0} := \{1+y \in 1+\widehat{I}_{L_0} | \widehat{U\Delta}(1+y) = (1+y)\widehat{\otimes}1 + 1\widehat{\otimes}(1+y)\}$.

**Lemma.**

$$P_{L_0} \underset{\log}{\overset{\exp}{\rightleftarrows}} G_{L_0},$$

are inverse bijections.
**Proposition.** $L_0 \to P_{L_0}$ is injective. If in addition $\dim H^1(\Lambda V) < \infty$, then it is an isomorphism.
**Proposition.** The restriction of the multiplication of $\widehat{UL_0}$ to $G_{L_0}$ gives the latter a group structure.

## Definition

$\pi_1(\Lambda V)$ is defined as the group $G_{L_0}$ when $\dim H^1(\Lambda V) < \infty$.

# Realization

Given a Sullivan algebra $\Lambda V$ (or more generally any CDGA), we can construct a simplicial set $\langle \Lambda V \rangle$, such that

$$\langle \Lambda V \rangle_n = \{\text{CDGA morphisms } \Lambda V \to (A_{PL})_n\}.$$

For $\sigma \in \langle \Lambda V \rangle_n$, $\partial_i(\sigma) = \partial_i \circ \sigma$ and $s_j(\sigma) = s_j \circ \sigma$.
For $\phi : \Lambda V \to \Lambda W$, there exists a simplicial map $\langle \phi \rangle : \langle \Lambda W \rangle \to \langle \Lambda V \rangle$, $\sigma \mapsto \sigma \circ \phi$.
In summary, $\langle - \rangle$ is a contravariant functor **CDGA** $\to$ **sSet**.

**Milnor realization** is a functor $| - | :$ **sSet**$\to$**Top** such that

$$|\Lambda V| = (\coprod_n \langle \Lambda V \rangle_n \times \Delta^n)/\sim,$$

$$(\partial_i \sigma, x) \sim (\sigma, \lambda_i x), \quad (s_j \sigma, x) \sim (\sigma, \rho_j x),$$

$$\lambda_i : \Delta^{n-1} \to \Delta^n,\ (a_0, \ldots, a_{n-1}) \mapsto (a_0, \ldots, a_{i-1}, 0, a_i, \ldots, a_{n-1}),$$

$$\rho_j : \Delta^{n+1} \to \Delta^n,\ (a_0, \ldots, a_{n+1}) \mapsto (a_0, \ldots, a_{j-1}, a_j + a_{j+1}, a_{j+2}, \ldots, a_{n+1}).$$

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# $\langle \Lambda V \rangle$ and $|\Lambda V|$

**Fact.** $|\Lambda V|$ is a CW complex. Its *n*-cells are identified with the non-degenerate *n*-simplices. This leads to a quasi-isomorphism

$$C_*(\langle \Lambda V \rangle) \xrightarrow{\simeq} C_*(|\Lambda V|).$$

Then

$$C^*(|\Lambda V|) \xrightarrow{\simeq} C^*(\langle \Lambda V \rangle) \quad \text{and} \quad A_{PL}(|\Lambda V|) \xrightarrow{\simeq} A_{PL}(\langle \Lambda V \rangle).$$

**Proposition**

*$A_{PL}(|\Lambda V|) \xrightarrow{\simeq} A_{PL}(\langle \Lambda V \rangle)$ is a surjective quasi-isomorphism.*

**Idea of proof.**
$\langle \Lambda V \rangle \to S_*(|\Lambda V|)$ is injective. Also all $A^n_{PL}$ are *extendable* (Any simplicial map $\partial \Delta^k \to A^n_{PL}$ can be extended to $\Delta^k$).

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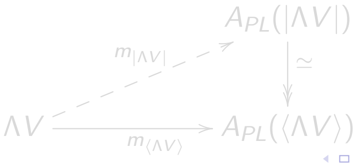
# Morphism $m_{|\Lambda V|} : \Lambda V \to A_{PL}(|\Lambda V|)$

Recall that $\langle \Lambda V \rangle_n$ consists of morphisms from $\Lambda V$ to $(A_{PL})_n$. Taking adjoint induces a CDGA morphism $m_{\langle \Lambda V \rangle}$.

$$id : \langle \Lambda V \rangle \to \langle \Lambda V \rangle$$
$$\Downarrow$$
$$\langle \Lambda V \rangle \times \Lambda V \to A_{PL}$$
$$\Downarrow$$
$$m_{\langle \Lambda V \rangle} : \Lambda V \to A_{PL}(\langle \Lambda V \rangle) = \{\text{simplicial maps } \langle \Lambda V \rangle \to A_{PL}\}$$

For example, given $x \in \Lambda V$ and $\sigma \in \langle \Lambda V \rangle$, $[m_{\langle \Lambda V \rangle}(x)](\sigma) = \sigma(x) \in A_{PL}$.

Lift $m_{\langle \Lambda V \rangle}$ through $A_{PL}(|\Lambda V|) \to A_{PL}(\langle \Lambda V \rangle)$, we obtain a morphism $m_{|\Lambda V|}$ uniquely up to homotopy.



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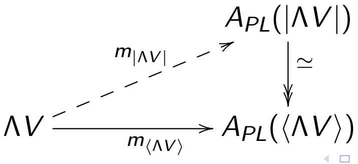

# Realization preserves homotopy

**Fact.** $\pi_n(\Lambda V) \cong \pi_n(|\Lambda V|)$ when $\dim H^1(\Lambda V) < \infty$.

Let $\alpha \in \pi_n(|\Lambda V|)$ with a representative $\sigma : S^n \to |\Lambda V|$. Take $m(n) : \Lambda W \to A_{PL}(S^n)$ as a minimal Sullivan model. There exists a lift $\chi : \Lambda V \to \Lambda W$ (uniquely up to homotopy) making the following diagram homotopy commutative.

$$\begin{array}{ccc} \Lambda V & \overset{\chi}{\dashrightarrow} & \Lambda W \\ {\scriptstyle m_{|\Lambda V|}}\downarrow & & \downarrow {\scriptstyle m(n)}\ \simeq \\ A_{PL}(\Lambda V) & \underset{A_{PL}(\sigma)}{\longrightarrow} & A_{PL}(S^n) \end{array}$$

Recall that homotopic maps induce the same morphism $V \cong Q(\Lambda V) \to Q(\Lambda W) \cong W$. So $\chi$ induces a morphism $Q(\chi) : V \to W$ independent on the choices of $\sigma$ and $\chi$.

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Let $w \in W^n$ such that $(m(n))w$ represents the fundamental class of $S^n$. Set $\alpha_f \in \mathrm{Hom}(V^n, \mathbb{Q})$ such that

$$\alpha_f(v)w = (Q(\chi))v.$$

Define

$$\iota_n : \pi_n(|\Lambda V|) \to \pi_n(\Lambda V), \quad \alpha \mapsto \begin{cases} \alpha_f, & n \geq 2; \\ \exp \alpha_f, & n = 1. \end{cases}$$

**Theorem**

*$\iota_n$ is an isomorphism of groups.*

**Idea of proof.**

Bijectivity. Construct an inverse map $\tau_n : \pi_n(\Lambda V) \to \pi_n(|\Lambda V|)$.
Consider the simplicial set $\Delta[n]/\partial\Delta[n]$. It has two non-degenerated simplices, $c_0$ of degree 0 and $c_n$ of degree $n$.
For $f \in \pi_n(\Lambda V) = \mathrm{Hom}(V, \mathbb{Q})$. Set

$$\phi_f : \Lambda V \to A_{PL}(\Delta[n]/\partial\Delta[n]),$$
$$v \mapsto \begin{pmatrix} c_n \mapsto (-1)^n f(v) dt_1 \wedge \ldots \wedge dt_n \in (A_{PL})^n_n \\ c_0 \mapsto 0 \in (A_{PL})^n_0 = 0 \end{pmatrix}.$$

Taking its adjoint gives

$$\langle \phi_f \rangle : \Delta[n]/\partial\Delta[n] \to \langle \Lambda V \rangle.$$

And the realization gives

$$|\phi_f| : S^n = |\Delta[n]/\partial\Delta[n]| \to |\Lambda V|.$$

Set $\tau_n(f)$ as the homotopy class of $|\phi_f|$.

Group morphism. Use the quotient map $\Lambda V \to \Lambda V^{\geq n}$, the inclusion $\Lambda V^n \to \Lambda V^{\geq n}$, and the naturality of $\iota_n$. The problem can be reduced to proving that $\iota_n : \pi_n(|\Lambda V^n|) \to \pi_n(\Lambda V^n)$ is an isomorphism.

When $n \geq 2$, this can be obtained by realizing the diagonal map

$$\Delta : \Lambda V^n \to \Lambda(V^n \oplus V^n) = \Lambda V^n \otimes \Lambda V^n,$$

which gives

$$|\Delta| : |\Lambda V^n \otimes \Lambda V^n| = |\Lambda V^n| \times |\Lambda V^n| \to |\Lambda V^n|.$$

The induced map on $\pi_n$ then shows that $\iota_n$ is a group morphism.

The case $n = 1$ is more complicated as $\pi_1$ is non-abelian. We need to construct a homotopy between the representatives.

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## Realization of relative Sullivan algebras

**Proposition**

1. *Given a relative Sullivan algebra* $(\Lambda V \otimes \Lambda Z, d)$, *the realizations* $|\Lambda Z| \to |\Lambda V \otimes \Lambda Z| \to |\Lambda V|$ *form a fiber bundle.*
2. *Given a product of Sullivan algebras* $(\Lambda V, d) \otimes (\Lambda Z, d)$, *its realization is* $|\Lambda V| \times |\Lambda Z|$.

**Idea of proof.**

First show that $\langle \Lambda Z \rangle \to \langle \Lambda V \otimes \Lambda Z \rangle \to \langle \Lambda V \rangle$ form a simplicial fiber bundle.

**Theorem**

*The Lie bracket on the homotopy Lie algebra $L = s^{-1}Hom(V, \mathbb{Q})$ of $\Lambda V$ is same as the Whitehead product on $\pi_*(|\Lambda V|)$ up to sign.*
*For $\alpha \in \pi_1(|\Lambda V|), \beta \in \pi_n(|\Lambda V|)$,*

$$\iota_n(\beta \bullet \alpha) = Ad(\iota_1(\alpha))^{-1}(\iota_n(\beta)),$$

*where $Ad(\exp_{L_0} x)(y) = e^{ad\, x}(y)$ and $(ad\, x)(y) = [x, y]$ for $x, y \in L$.*
*In particular, when $n = 1$,*

$$\iota_1(\beta \bullet \alpha) = \iota_1(\alpha^{-1})\iota_1(\beta)\iota_1(\alpha).$$

# When $m_{|\Lambda V|}$ is a quasi-isomorphism

**Theorem**

*$m_{|\Lambda V|}$ is a quasi-isomorphism if $\Lambda V$ is simply-connected ($V^1 = 0$) and of finite type, i.e. such $\Lambda V$ is a model of its realization.*

**Remark.** When $\Lambda V$ is simply-connected,
$H^*(\Lambda V)$ is of finite type $\Longrightarrow \Lambda V$ has finite type.

**Idea of proof.**
1. Reduce to the case $\Lambda V$ being minimal.
$\Lambda V$ can be written as $\Lambda W \otimes \Lambda(U \oplus dU)$, where $\Lambda W$ is minimal. Then $H^*(\Lambda W) = H^*(\Lambda V)$. On the other hand, it can be proved that $|\Lambda(U \oplus dU)|$ is contractible.

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2. Consider the case $V = V^n$.
Then $|\Lambda V| = K(\mathbb{Q}^{\dim V}, n)$. Let $\Lambda W$ be the minimal Sullivan model. As proved earlier, $W = W^n$ and $\pi_n(|\Lambda V|) = \mathrm{Hom}(W^n, \mathbb{Q})$. So $V^n \cong W^n$ and $\Lambda V \cong \Lambda W$.

3. Consider the case $V = V^{\leq n}$.
Use induction. Suppose that the statement holds for $n-1$. Then there exists the following commutative diagram.

$$\begin{array}{ccccc} \Lambda V^{<n} & \hookrightarrow & \Lambda V = \Lambda V^{<n} \otimes \Lambda V^n & \xrightarrow{pr} & \Lambda V^n \\ \downarrow \simeq & & \downarrow & & \downarrow \simeq \\ A_{PL}(|\Lambda V^{<n}|) & \longrightarrow & A_{PL}(|\Lambda V|) & \longrightarrow & A_{PL}(|\Lambda V^n|) \end{array}$$

So the middle vertical map is also a quasi-isomorphism.

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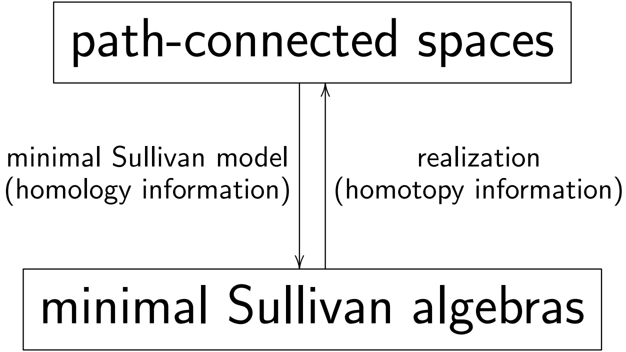
So the middle vertical map is also a quasi-isomorphism.

4. Consider the general case.
For each fixed $n$, $\Lambda V = \Lambda V^{\leq n+1} \otimes \Lambda V^{>n+1}$. The composition of $m_{|\Lambda V^{\leq n+1}|} : \Lambda V^{\leq n+1} \to A_{PL}(|\Lambda V^{\leq n+1}|)$ and $A_{PL}(|\Lambda V^{\leq n+1}|) \to A_{PL}(|\Lambda V|)$ can be factored through some minimal relative Sullivan algebra $\Lambda V^{\leq n+1} \otimes \Lambda W$ quasi-isomorphic to $A_{PL}(|\Lambda V|)$. So we have the commutative diagram.

$$\begin{array}{ccccc}
\Lambda V^{\leq n+1} & \hookrightarrow & \Lambda V^{\leq n+1} \otimes \Lambda W & \xrightarrow{pr} & \Lambda W \\
\simeq \downarrow & & \simeq \downarrow & & \downarrow \\
A_{PL}(|\Lambda V^{\leq n+1}|) & \longrightarrow & A_{PL}(|\Lambda V|) & \longrightarrow & A_{PL}(|\Lambda V^{>n+1}|)
\end{array}$$

Then $\Lambda W$ is a minimal Sullivan model of $|\Lambda V^{>n+1}|$. As $|\Lambda V^{>n+1}|$ is $(n+1)$-connected, we have $W = W^{>n+1}$. Thus,

$$H^n(|\Lambda V|) = H^n(\Lambda V^{\leq n+1} \otimes \Lambda W) = H^n(\Lambda V^{\leq n+1}) = H^n(\Lambda V).$$




When both objects are simply-connected and have finite type (rational) cohomology, the two functors are inverse to each other, up to equivalence.

**Equivalence for spaces:** connected by maps preserving rational cohomology (also rational homotopy when simply-connected).
**Equivalence for minimal Sullivan algebras:** isomorphism.

# Infinite dimensional case

**Question.** When does the realization preserve cohomology?

**Theorem**

*Let $\Lambda V$ be a simply connected minimal Sullivan algebra. Then $\Lambda V$ is a model of $|\Lambda V|$ if and only if $V$ has finite type.*

**Idea of proof.**
If $\dim V^2 = \infty$, the dimension of

$$H_2(|\Lambda V|; \mathbb{Q}) = \pi_2(|\Lambda V|) = \pi_2(\Lambda V) = \mathrm{Hom}(V^2, \mathbb{Q})$$

has a larger cardinality than $\dim V^2$. As $H^2(|\Lambda V|; \mathbb{Q})$ is the dual of $H_2(|\Lambda V|; \mathbb{Q})$, its dimension has an even larger dimension. So

$$H^2(|\Lambda V|; \mathbb{Q}) \neq H^2(\Lambda V).$$

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For the general case, consider the smallest $n$ such that $\dim V^n = \infty$.

## Proposition (Z. 2024)

*Let $\Lambda V$ be a minimal Sullivan algebra. If some $H^n(\Lambda V)$ is infinite dimensional, then $\Lambda V$ cannot be a model of $|\Lambda V|$.*

**Remark.** This include the case that dim $V^n$ is uncountable for some $n$. The case that $V$ is not of finite type but $H^*(\Lambda V)$ is is more complicated. When such $\Lambda V$ is a model of $|\Lambda V|$ remains open.

# Sullivan spaces

**Definition**

A **Sullivan space** $X$ is a path-connected space $X$ such that
(i) $\dim H^1(X;\mathbb{Q}),\ \dim H^k(\widetilde{X};\mathbb{Q}) < \infty$ for $k \geq 2$, where $\widetilde{X}$ is the universal cover of $X$.
(ii) $\pi_k(X) \otimes \mathbb{Q} \cong \pi_k(\Lambda V)$ for $k \geq 2$, where $\Lambda V$ is the minimal Sullivan model of $X$.

# Properties of Sullivan spaces

Consider the following spaces.

$X$: a connected CW complex.

$B = B\pi_1(X)$: the classifying space of $\pi_1(X)$.

$MB$: the space of (Moore) paths on $B$.

$X \times_B MB$: the fiber product of the canonical $X \to B$ which is identity on $\pi_1$ and the fibration $MB \to B$. It is homotopy equivalent to $X$.

$F$: the fiber of $X \times_B MB \to B$. It is simply connected and weak homotopy equivalent to $\widetilde{X}$.

Applying Serre spectral sequence, we have that $H^1(B) \to H^1(X \times_B MB)$ is isomorphic and $H^2(B) \to H^2(X \times_B MB)$ is injective.

Take $\Lambda V^1 \otimes \Lambda W^{\geq 2}$ the minimal Sullivan model of $B$. Then it can be extended to a model $\Lambda V^1 \otimes \Lambda W^{\geq 2} \otimes \Lambda Z^{\geq 2}$ of $X \times_B MB$. Write this model as $\Lambda V \otimes \Lambda(U \oplus dU)$ (For degree reason, $V^1 \subset V$).

This induces the following commutative diagram.

$$\begin{array}{ccccc}
\Lambda V^1 & \hookrightarrow & \Lambda V & \xrightarrow{pr} & \Lambda V^{\geq 2} \\
\downarrow{\scriptstyle m_B} & & \downarrow{\scriptstyle m_X}\,\simeq & & \downarrow{\scriptstyle m_F} \\
A_{PL}(B) & \longrightarrow & A_{PL}(X \times_B MB) & \longrightarrow & A_{PL}(F)
\end{array}$$

## Theorem

*When $X$ is a Sullivan space, the following statement holds.*
*(1) $m_B$ is a quasi-isomorphism and $H^*(B)$ has finite type.*
*(2) $m_F$ is a quasi-isomorphism.*
*(3) $\dim H^1(X) < \infty$ and $H^*(\widetilde{X})$ has finite type.*
*(4) $H^*(X)$ has finite type.*
*(5) $\pi_1(X)$ acts on each $H^k(\widetilde{X})$ nilpotently via covering transformations.*

*Conversely, if $m_B$ is a quasi-isomorphism, then (2)(3), (3)(5) or (4)(5) implies that $X$ is a Sullivan space.*

# When the realization is a Sullivan space

**Theorem (c.f. *Rational Homotopy Theory II*)**

*For any minimal Sullivan algebra $\Lambda V$, the following conditions are equivalent.*
*(i)* $\dim H^1(\Lambda V)$ *and* $\dim V^i$ *for* $i \geq 2$ *are finite dimensional, and the canonical morphism* $m_{|\Lambda V|} : \Lambda V \to A_{PL}(|\Lambda V|)$ *is a quasi-isomorphism.*
*(ii)* $|\Lambda V|$ *is a Sullivan space.*

**Open problem(Félix-Halperin-Thomas).**
If $m_{|\Lambda V|}$ is a quasi-isomorphism, must $|\Lambda V|$ be a Sullivan space?

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**Idea of proof.**

## Proposition

*Suppose that $\Lambda W^1$ is a minimal Sullivan algebra. Then* $\dim H^1(\Lambda W^1) < \infty$ *if and only if* $H^1(|\Lambda W^1|) < \infty$.
*In this case,* $m_{|\Lambda W^1|} : \Lambda W^1 \to A_{PL}(|\Lambda W^1|)$ *extends to a minimal Sullivan model*

$$\Lambda W^1 \otimes \Lambda Z^{\geq 2} \xrightarrow{\simeq} A_{PL}(|\Lambda W^1|),$$

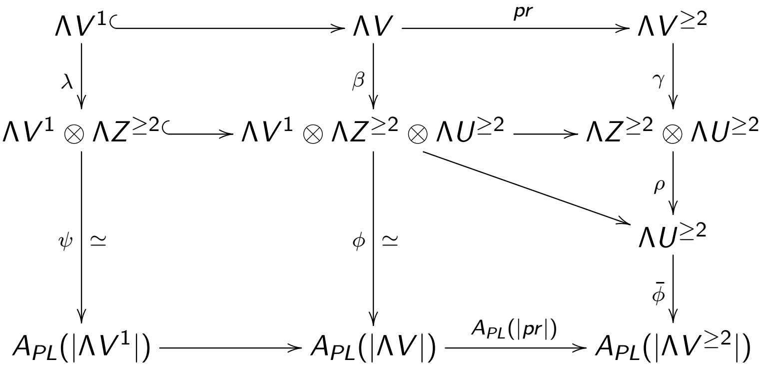
*and* $\dim H^1(\Lambda W^1) = H^1(|\Lambda W^1|)$.

By this proposition, that either (i) or (ii) holds implies $\dim H^1(\Lambda V) < \infty$. So the minimal model of $|\Lambda V^1|$ is of the form $\Lambda V^1 \otimes \Lambda Z^{\geq 2}$. This gives a minimal relative Sullivan algebra $\Lambda V^1 \otimes \Lambda Z^{\geq 2} \otimes \Lambda U^{\geq 2}$ which is a model of $|\Lambda V|$, and the following commutative diagram.

$$\begin{array}{ccccc}
\Lambda V^1 \otimes \Lambda Z^{\geq 2} & \hookrightarrow & \Lambda V^1 \otimes \Lambda Z^{\geq 2} \otimes \Lambda U^{\geq 2} & \xrightarrow{pr} & \Lambda U^{\geq 2} \\
\psi \downarrow \simeq & & \phi \downarrow \simeq & & \downarrow \bar{\phi} \\
A_{PL}(|\Lambda V^1|) & \longrightarrow & A_{PL}(|\Lambda V|) & \longrightarrow & A_{PL}(|\Lambda V^{\geq 2}|)
\end{array}$$

$m_{|\Lambda V|}$ can be lifted through $\phi$ rel $\Lambda V^1$. That is, there exists a morphism $\beta : \Lambda V \to \Lambda V^1 \otimes \Lambda Z^{\geq 2} \otimes \Lambda U^{\geq 2}$ such that $\phi \circ \beta \sim m_{|\Lambda V|}$ and their restrictions to $\Lambda V^1$ are same.

This induces the commutative diagram below.



On the other hand, as $m_{\langle \Lambda V \rangle} : \Lambda V \to A_{PL}(\langle \Lambda V \rangle)$ is a natural transformation, the following diagram is commutative up to homotopy. This homotopy can be made rel $\Lambda V^1$.

$$\begin{array}{ccc} \Lambda V & \xrightarrow{\quad pr \quad} & \Lambda V^{\geq 2} \\ \Big\downarrow{\scriptstyle m_{|\Lambda V|}} & & \Big\downarrow{\scriptstyle m_{|\Lambda V^{\geq 2}|}} \\ A_{PL}(|\Lambda V|) & \xrightarrow{A_{PL}(|pr|)} & A_{PL}(|\Lambda V^{\geq 2}|) \end{array}$$

Thus,

$$m_{|\Lambda V^{\geq 2}|} \circ pr \sim \bar{\phi} \circ \rho \circ \gamma \circ pr \text{ rel } \Lambda V^1.$$

This leads to

$$m_{|\Lambda V^{\geq 2}|} \sim \bar{\phi} \circ \rho \circ \gamma.$$

Now suppose (i) holds. It is sufficient to show that the commutative diagram below satisfies the following statements of the previous theorem.

$$
\begin{array}{ccccc}
\Lambda V^1 & \hookrightarrow & \Lambda V & \xrightarrow{pr} & \Lambda V^{\geq 2} \\
\downarrow{\scriptstyle m_{|\Lambda V^1|}} & & \downarrow{\scriptstyle m_{|\Lambda V|}} & & \downarrow{\scriptstyle m_{|\Lambda V^{\geq 2}|}} \\
A_{PL}(|\Lambda V^1|) & \longrightarrow & A_{PL}(|\Lambda V|) & \longrightarrow & A_{PL}(|\Lambda V^{\geq 2}|)
\end{array}
$$

(1) $m_{|\Lambda V^1|}$ is a quasi-isomorphism.
(2) $m_{|\Lambda V^{\geq 2}|}$ is a quasi-isomorphism.
(3) $\dim H^1(|\Lambda V|) < \infty$ and $H^*(|\Lambda V^{\geq 2}|)$ has finite type.

By hypothesis $V^{\geq 2}$ has finite type. So $m_{|\Lambda V^{\geq 2}|}$ is a quasi-isomorphism. This proves (2).

It follows that $H^*(|\Lambda V^{\geq 2}|) = H^*(\Lambda V^{\geq 2})$ is of finite type. Also by hypothesis $m_{|\Lambda V|}$ is a quasi-isomorphism. So $\dim H^1(|\Lambda V|) = \dim H^1(\Lambda V) < \infty$. This proves (3).

**Fact.** $\pi_1(\Lambda V^1)$ acting on $H^*(\Lambda V^{\geq 2})$ is locally nilpotent. Moreover, this action can be identified with $\pi_1(|\Lambda V^1|)$ acting on the image of $H^*(\Lambda V^{\geq 2}) \to H^*(|\Lambda V^{\geq 2}|)$ induced by $m_{|\Lambda V^{\geq 2}|}$.

Now that $m_{|\Lambda V^{\geq 2}|}$ is a quasi-isomorphism, this image is just $H^*(|\Lambda V^{\geq 2}|)$. Thus, that $\psi$ and $\phi$ are quasi-isomorphisms implies that so is $\bar{\phi}$.

Then $m_{|\Lambda V^{\geq 2}|} \sim \bar{\phi} \circ \rho \circ \gamma$ implies that $\rho \circ \gamma$ is also a quasi-isomorphism. On the other hand, $\phi \circ \beta \sim m_{|\Lambda V|}$ implies that $\beta$ is a quasi-isomorphism. These lead to that $\lambda$ is a quasi-isomorphism. In particular, $Z = 0$.

By construction, $m_{|\Lambda V^1|} = \phi \circ \lambda$. So it is a quasi-isomorphism. This proves (1).

Suppose (ii) holds. By the Properties of Sullivan spaces, $H^*(|\Lambda V^{\geq 2}|)$ are of finite type. Then so are $\pi_*(|\Lambda V^{\geq 2}|)$ and $V^{\geq 2}$.

That $V^{\geq 2}$ is of finite type also implies that $m_{|\Lambda V^{\geq 2}|}$ is a quasi-isomorphism. Moreover, recall that the minimal model of $|\Lambda V^1|$ is of the form $\Lambda V^1 \otimes \Lambda Z^2$. $|\Lambda V|$ being a Sullivan space implies that $Z = 0$ and $m_{|\Lambda V^1|}$ is a quasi-isomorphism. It also implies that $\pi_1(|\Lambda V^1|)$ acts on $H^*(|\Lambda V^{\geq 2}|)$ nilpotently. Therefore, $m_{|\Lambda V|}$ is a quasi-isomorphism.

Finally, $|\Lambda V|$ being a Sullivan space also implies $\dim H^1(|\Lambda V|) < \infty$. Together with $m_{|\Lambda V|}$ being a quasi-isomorphism, we have that $\dim H^1(\Lambda V) < \infty$.

# A Theorem for classifying space

**Theorem**

*If the classifying spaces $BG_1$ and $BG_2$ are Sullivan spaces, then so is the classifying space $B(G_1 * G_2)$ of the free product.*

| $S^m \vee S^n$ | Sullivan space? |
|---|---|
| $m, n > 1$ | Yes |
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# Formality

**Definition**

A CDGA is called **formal** if its cohomology ring serves as its model.
A topological space $X$ is called **formal** if $A_{PL}(X)$ is a formal CDGA.

**Theorem (Sullivan 1977; Halperin-Stasheff, 1979)**

*$A_{PL}(X)$ is formal if and only if the tensor product of $A_{PL}(X)$ and some extension field over $\mathbb{Q}$ is formal.*

Thus, a smooth manifold $M$ is formal if and only if $\Omega^*(M)$ is formal.
(**Remark.** In general, equivalent CDGA over $\mathbb{R}$ may not be equivalent over $\mathbb{Q}$.)

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# Examples of formal spaces.

**Theorem (Deligne-Griffiths-Morgan-Sullivan, 1975)**

*A complex manifold where the $dd^c$-lemma holds is formal. In particular, all compact Kähler manifolds are formal.*

**Theorem (Miller, 1979)**

*Let $X$ be an $n$-dimensional $k$-connected compact space. If $n \leq 4k+2$, then $X$ is formal.*

Other formal spaces include Lie groups, H-spaces, homogeneous spaces, product of formal spaces, and wedge sum of formal spaces.

Non-formal spaces include nilmanifolds except torus.

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*A complex manifold where the $dd^c$-lemma holds is formal. In particular, all compact Kähler manifolds are formal.*

**Theorem (Miller, 1979)**

*Let $X$ be an $n$-dimensional $k$-connected compact space. If $n \leq 4k + 2$, then $X$ is formal.*

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# Questions about formality of sphere bundles

Let $X$ be an orientable $S^k$-bundle over $M$. We naturally ask the relationship between their formalities:

1. If $M$ is formal, when is $X$ formal?
2. What properties of $M$ are corresponding to the formality of $X$?

**Example**

Let $M = T^2$ and $X$ be a non-trivial orientable circle bundle over $M$. Then $M$ is formal but $X$ is non-formal.

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Let $M = S^2$ and $X$ be an arbitrary circle bundle over $M$. Then both $M$ and $X$ are formal.

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# About Question 1

**Theorem (Z, 2019)**

*If $M$ is formal, then $\Omega^*(X)$ has an $A_\infty$-minimal model whose only non-trivial operations are $m_2$ and $m_3$.*

If an $A_\infty$-algebra has an $A_\infty$-minimal model with only $m_2$ non-trivial, then it is **formal**.

**Theorem (Crowley-Nordström, 2020)**

*If $X$ is a compact manifold, and the Bianchi-Massey tensor of $\Omega^*(X)$ vanishes, then $\Omega^*(X)$ has an $A_\infty$-minimal model with $m_3 = 0$.*

**Conjecture.**

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# Bianchi-Massey Tensor

Let $E^*$ be the space of $e \in H^*(X) \otimes H^*(X)$ satisfying
1. $e$ is a graded symmetric tensor.
2. $e$ is in the kernel of the multiplication map $H^*(X) \otimes H^*(X) \to H^*(X)$.

Choose an arbitrary morphism (of graded vector spaces)

$$\alpha : H^*(X) \to \Omega^*(X)$$

sending cohomology classes to representatives. There exists a morphism

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of degree $-1$ satisfying $d\gamma = \alpha^2$, where $\alpha^2(x \otimes y) = \alpha(x) \wedge \alpha(y)$.

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On a subspace of $E^* \otimes E^*$ (graded symmetric tensors, in the kernel of full symmetrization), the degree $-1$ morphism

$$E^* \otimes E^* \to \Omega^*(X) \quad e \otimes e' \mapsto \gamma(e)\alpha^2(e')$$

takes closed values. Thus it induces a morphism from this subspace to $H^*(X)$, and this morphism is called the **Bianchi-Massey tensor**.

**Theorem (Crowley-Nordström, 2020)**

*The definition of Bianchi-Massey tensor is independent of the choices of $\alpha$ and $\gamma$.*

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# Main Theorem

**Theorem (Z, 2025)**

*Let $M$ be a compact formal manifold, and $X$ be an orientable $S^k$-bundle over $M$. Then $X$ is formal if and only if the Bianchi-Massey tensor of $\Omega^*(X)$ vanishes.*

*In particular, if $k$ is even, then $X$ is always formal.*

# A corollary about the unit tangent bundle

## Corollary (Z, 2025)

*Let $M$ be a compact orientable formal manifold. $X$ is a sphere bundle whose Euler class is the fundamental cohomology class. If $X$ is formal, then $H^*(M;\mathbb{R}) = \mathbb{R}[x]/(x^k)$ is a quotient of the polynomial ring with a single variable, i.e.*

$$H^*(M;\mathbb{R}) = \langle 1, x, x^2, \ldots, x^{k-1} \rangle.$$

## Corollary (Z, 2025)

*Let $M$ be a compact orientable formal manifold. Its unit tangent bundle $UTM$ is formal if and only if one of the following statement holds.*
*1. The Euler characteristic $\chi(M) = 0$.*
*2. $H^*(M;\mathbb{R}) = \mathbb{R}[x]/(x^k)$ is a quotient of the polynomial ring with a single variable.*

# Example of Riemann surfaces

Consider the circle bundle over a Riemann surface $M$. Let $[\omega]$ denote the fundamental cohomology class of $M$.

| genus | Euler class is $[\omega]$ | unit tangent bundle |
|---|---|---|
| 0 | formal | formal |
| 1 | non-formal | formal |
| $\geq 2$ | non-formal | non-formal |

# More General Base Manifold

**Theorem (Z, 2025)**

*Suppose $(M, \omega)$ is a symplectic manifold satisfying the hard Lefschetz property, and $X$ is a circle bundle over $M$ with Euler class $[\omega]$. If $[\omega]$ is reducible in $H^2(M)$, then $X$ cannot be formal.*

Here reducible means $[\omega] \in H^1(M) \cdot H^1(M)$, i.e. there exist $x_i, y_i \in H^1(M)$ such that

$$[\omega] = \sum x_i \wedge y_i.$$

Hard Lefschetz property means that for a $2n$-dimensional symplectic manifold, the following map is an isomorphism.

$$\omega^k : H^{n-k}(M) \xrightarrow{\cong} H^{n+k}(M), \quad x \mapsto [\omega^k] \wedge x.$$

# A Slightly Weaker Restriction

The hard Lefschetz property can be replaced by a weaker statement: Let $\omega \in H^{4r+2}(M)$, and there exists some $s \geq 0$ such that

1. $\omega : H^s(M) \to H^{s+4r+2}(M)$ is an isomorphism.
2. $\omega : H^{s-2r-1}(M) \to H^{s+2r+1}(M)$ is injective.

In this case the reducibility of $[\omega]$ is replaced by

$$[\omega] \in H^{2r+1}(M) \cdot H^{2r+1}(M).$$

Then the sphere bundle whose Euler class is $[\omega]$ is **non-formal**.

**Remark.**

That the degree of $\omega$ is $4r+2$ is necessary. There exist a counterexample for the case of $4r+4$. Let $M = \mathbb{C}P^2$ and $\omega \in \Omega^4(M)$ be its volume form. The above requirements are satisfied but the sphere bundle is formal.

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